Embedded Systems
2012/13


Basilica di Santa Maria di Collemaggio, 1287, L'Aquila

Lecture 4
Overview of Symbolic Models

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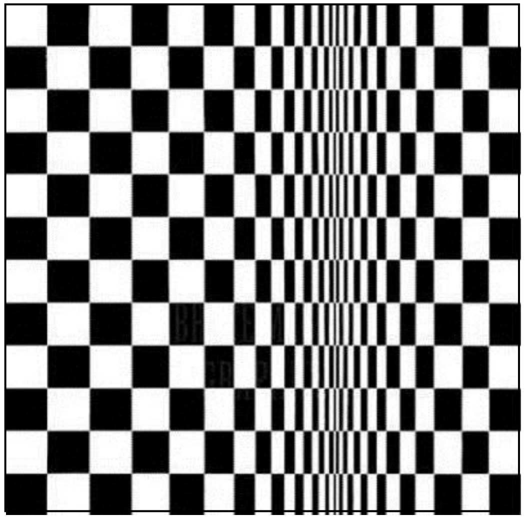
Continuous and Hybrid Systems

Salvador Dali, The Temptation of St. Anthony



Bridget Riley, Movement in Squares

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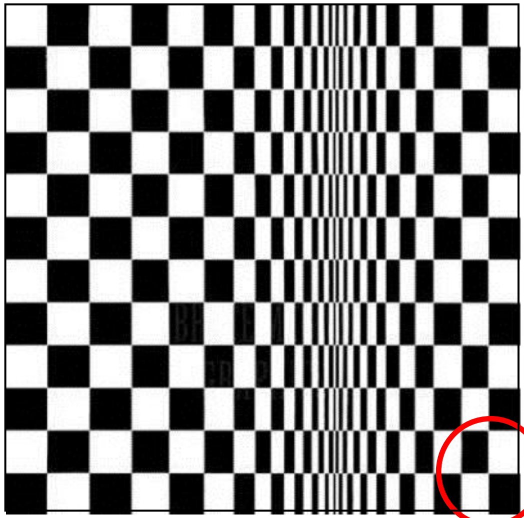

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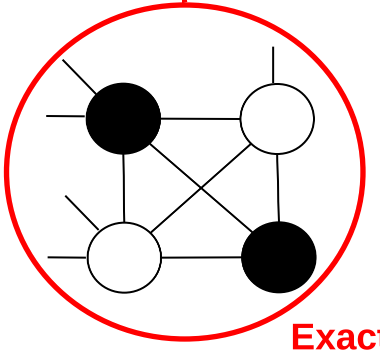

Exact

Continuous and Hybrid Systems

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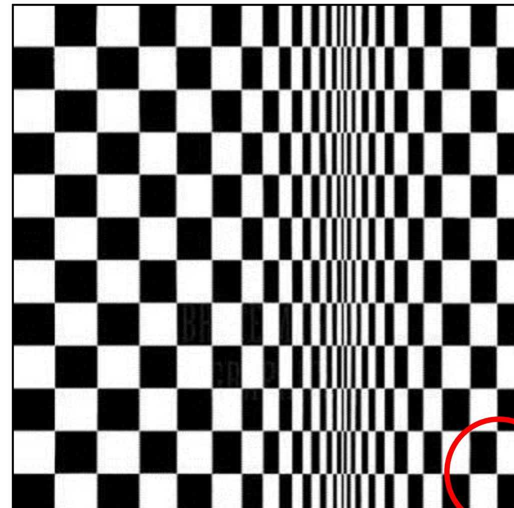
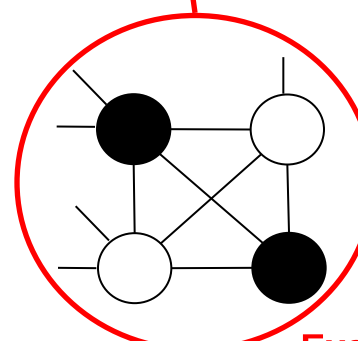


Bridget Riley, Movement in Squares

Exact ?



Exact

Continuous and Hybrid Systems

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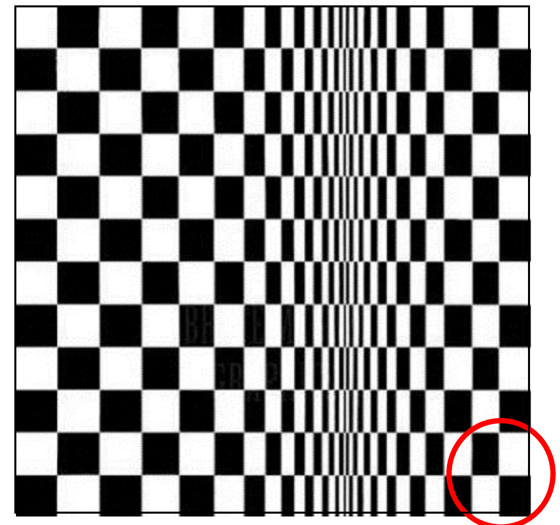

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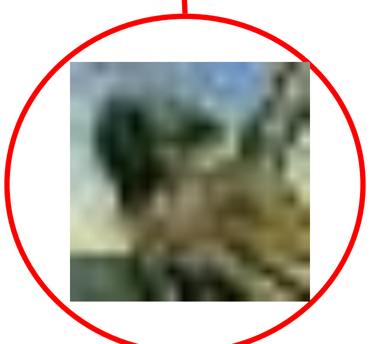
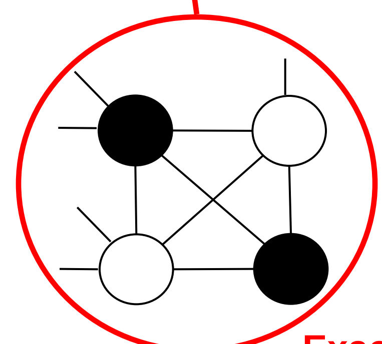


Approximated



Exact

Continuous and Hybrid Systems

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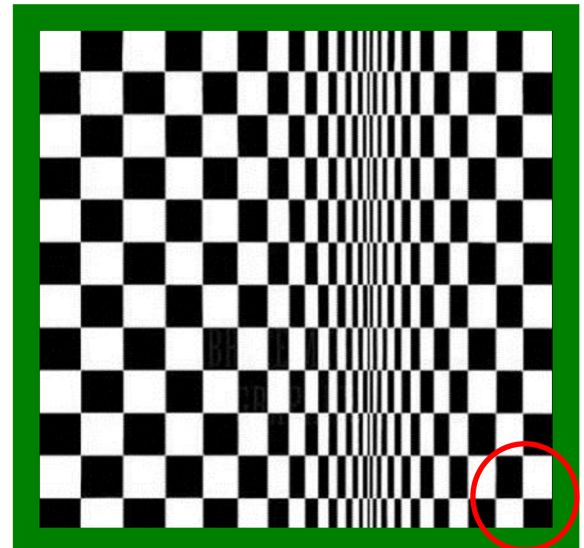

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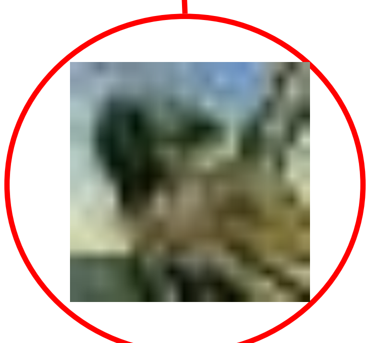
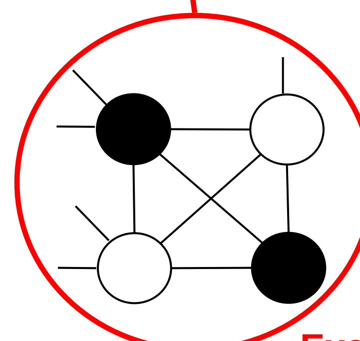


Approximated

There are systems that:

- Admit symbolic models
- Can be approximated by symbolic models
- Do not admit symbolic models



Exact

Continuous and Hybrid Systems

There are systems that:

- Admit symbolic models
  - Autonomous Systems (no input)
    - Timed Automata *[Alur and Dill, 1992]*
    - Multirate Automata *[Alur et al., 1993]*
    - Rectangular Automata *[Henzinger et al., 1998]*
    - o-minimal hybrid systems *[Lafferriere et al., 2000]*
  - Discrete-time linear control systems *[Tabuada et al., 2006]*
- Can be approximated by symbolic models
  - Incrementally stable nonlinear control systems *[Pola et al., 2008]*
  - Incrementally stable switched nonlinear systems *[Girard et al., 2008]*
  - Incrementally stable nonlinear control systems with disturbances *[Pola et al. 2008]*
- Do not admit symbolic models

Recall: Hybrid Systems

Definition 3.1 (Hybrid Automaton) *A hybrid automaton H is a collection $H = (Q, X, f, \text{I}nit, D, E, G, R)$, where*

- $Q = \{q_1, q_2, \ldots\}$ *is a set of* **discrete states***;*
- $X = \mathbb{R}^n$ *is a set of* **continuous states***;*
- $f(\cdot, \cdot) : Q \times X \to \mathbb{R}^n$ *is a* **vector field***;*
- $\text{I}nit \subseteq Q \times X$ *is a set of* **initial states***;*
- $\text{D}om(\cdot) : Q \to P(X)$ *is a* **domain***;*
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Recall that $P(X)$ denotes the power set (set of all subsets) of X. The notation of Definition 3.1 suggests, for example, that the function $\text{D}om$ assigns a set of continuous states $\text{D}om(q) \subseteq \mathbb{R}^n$ to to each discrete state $q \in Q$. We refer to $(q, x) \in Q \times X$ as the *state* of H.

Taken from: John Lygeros, Notes for an ENSIETA short course, February, 2004

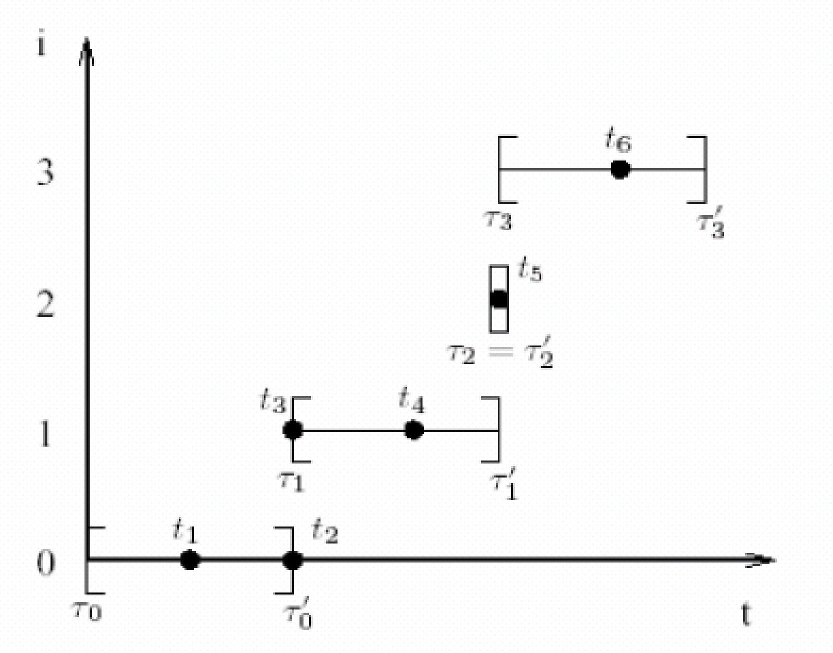
Base temporale

Base temporale (time basis): sequenza di $N+1$ intervalli $I_i = [\tau_i \, , \, \tau_i{}'] \; (i = 0, \ldots, N)$ tali che

- per ogni $i < N$, $\tau_i \leq \tau_i' = \tau_{i+1}$
- se $N < \infty$, $I_N = [\tau_N \, , \, \tau_N']$ o $I_N = [\tau_N \, , \, \tau_N')$

$\mathcal{T}$ denota l'insieme delle basi temporali

Recall: Hybrid Systems



Esecuzione

Una ***esecuzione*** di un sistema ibrido autonomo

$$H = (\ Q, X, Init, f, Dom, E, G, R\ )$$

è una ***traiettoria*** $\chi = (\tau\,, q\,, x)$

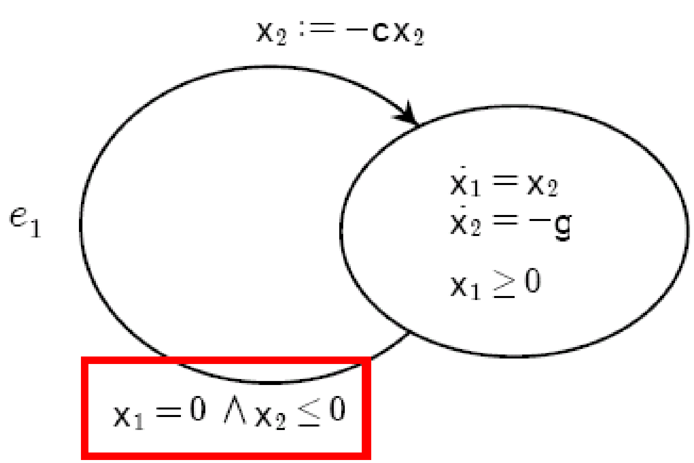
con $\tau \in \mathcal{T}$, $q : i \to Q$, $x : i \to x^i(t)$ tali che

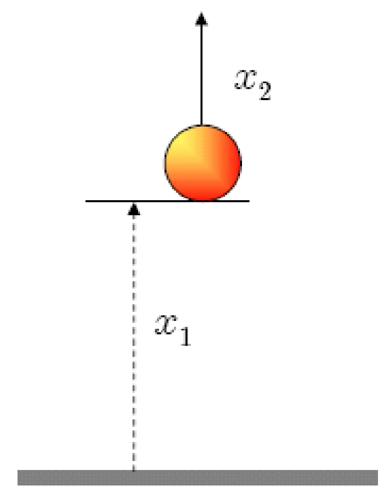
- $(\ q(0)\,, x^0(0)\ ) \in Init$
- evoluzione continua: $t \in [\tau_i\ , \tau_i{}')$, $\dot{x}^i(t) = f(q(i), x^i(t))$ e $x^i(t) \in Dom\,(q(i))$
- evoluzione discreta: per ogni $i = 0, \ldots, N - 1$,

$e = (q(i), q(i+1)) \in E$,

$x^i(\tau_i{}') \in G(e)$ e $x^{i+1}(\tau_{i+1}) \in R(e,\ x^i(\tau_i{}'))$

Recall: Hybrid Systems





Rectangular Automata

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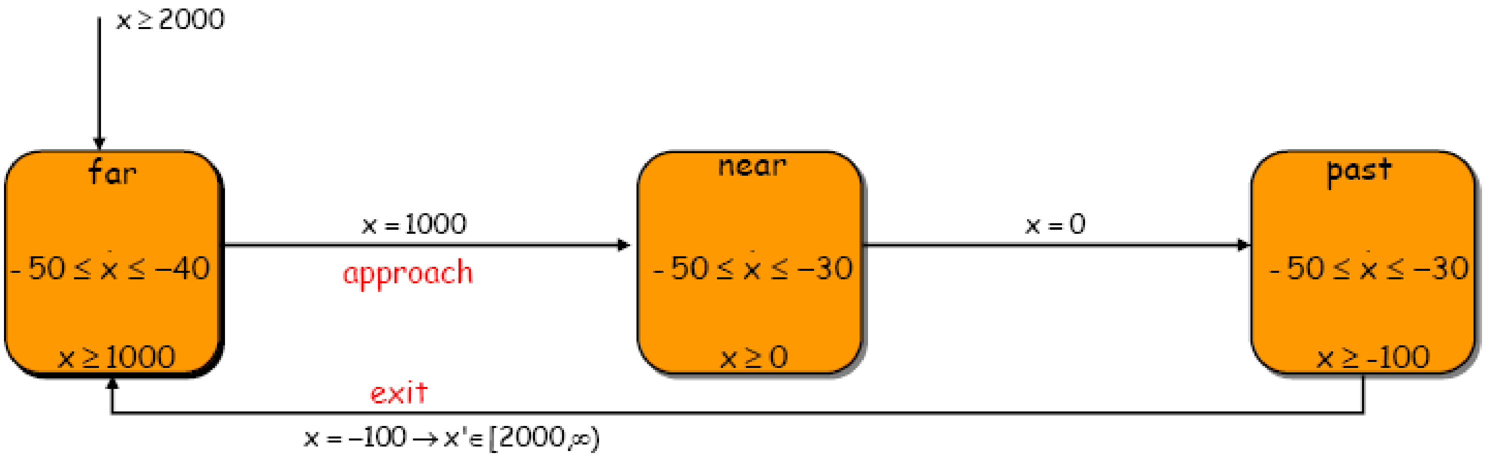
Taken from: John Lygeros, Notes for an ENSIETA short course, February, 2004

Rectangular Automata

Rectangular sets : $\bigwedge_i x_i \sim c_i \quad \sim \in \{<, \leq, =, \geq, >\}, c_i \in Q$



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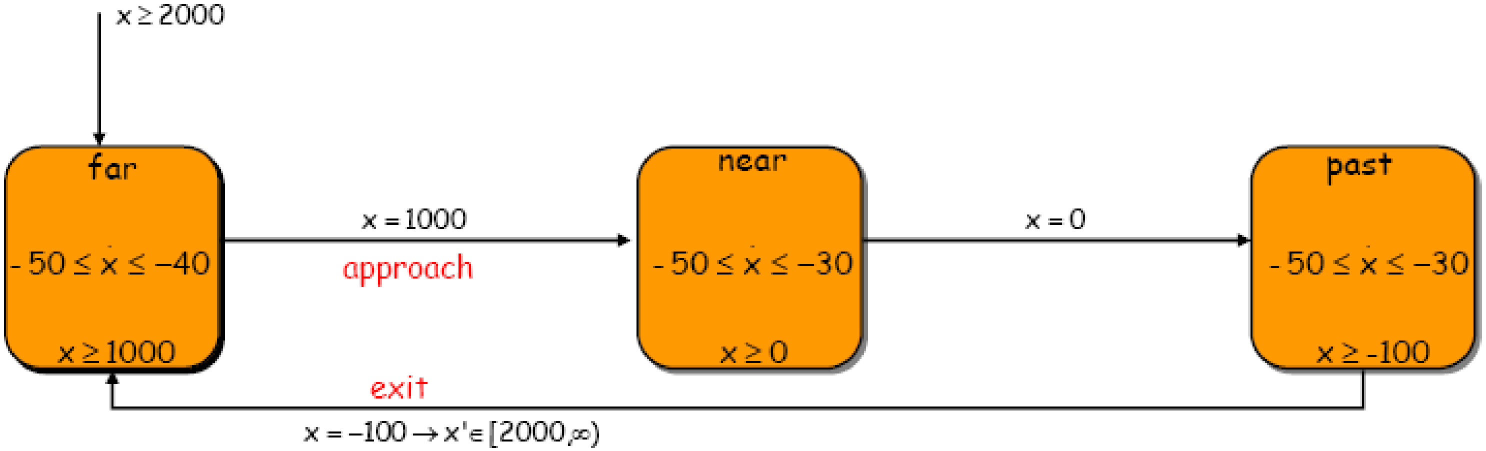

Rectangular hybrid automata are **initialized** if the following holds:

After a discrete transition, if the differential equation for a variable changes, then the variable must be reset to a fixed interval.

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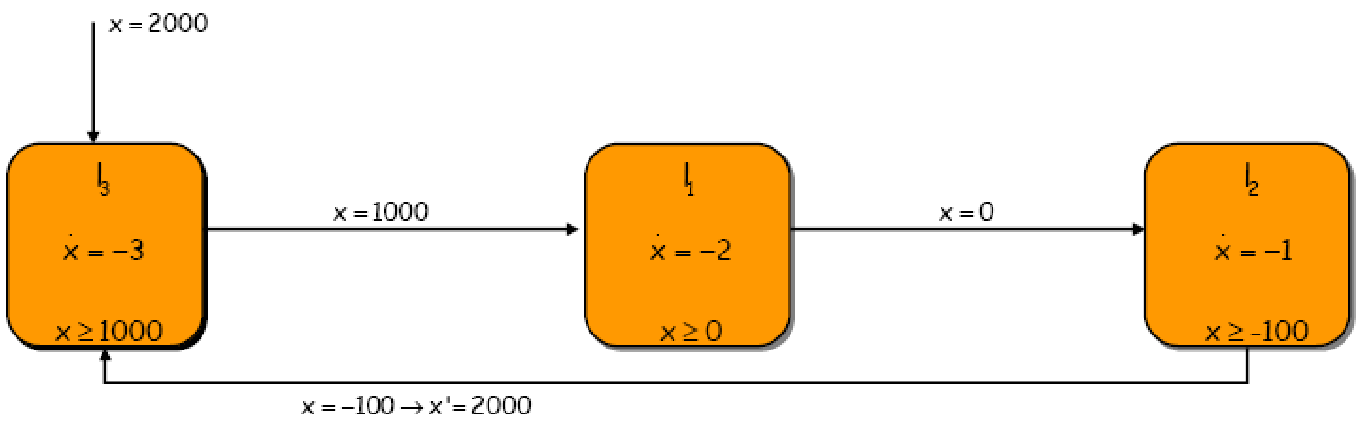

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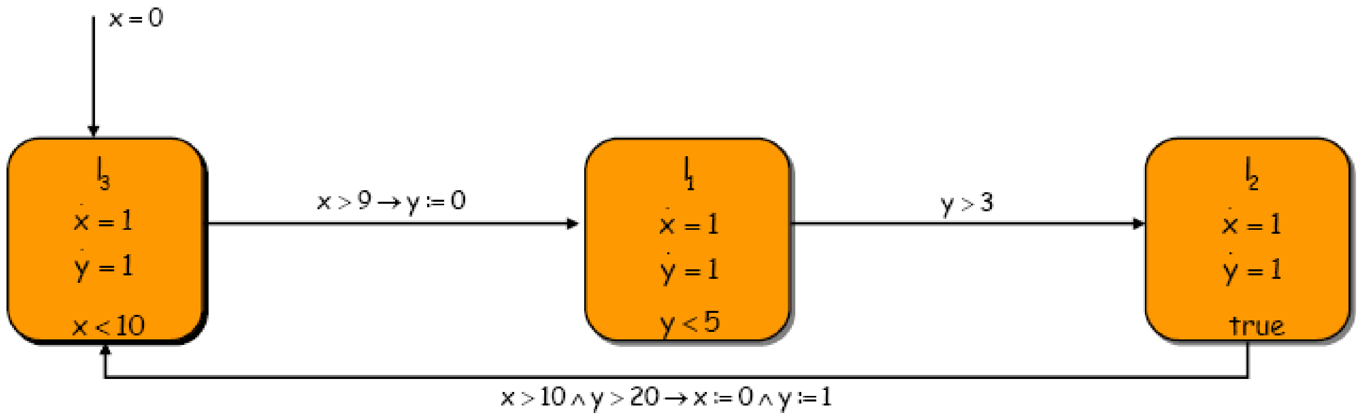
Timed Automata



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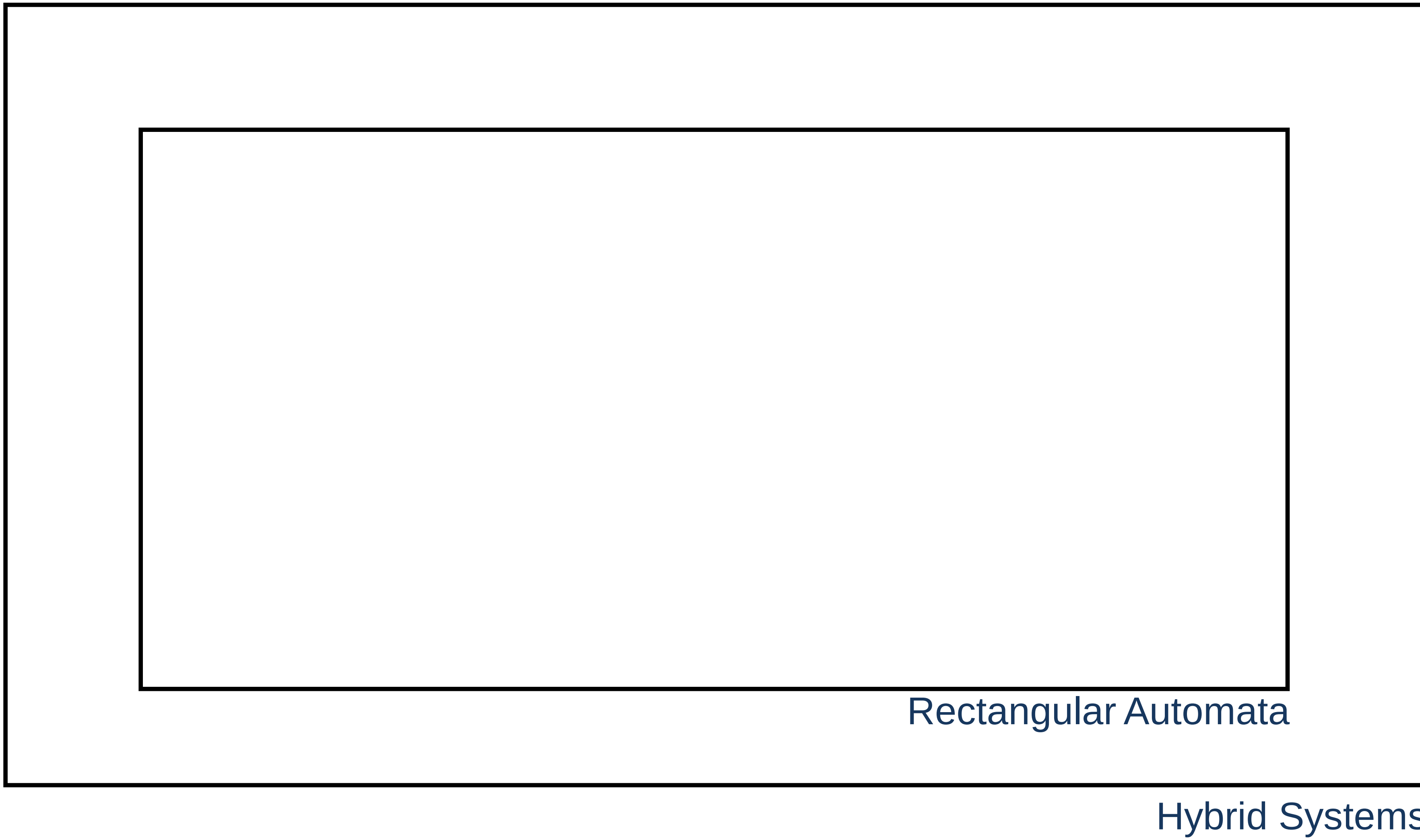
Summarizing …

Hybrid Systems

… for the time being…



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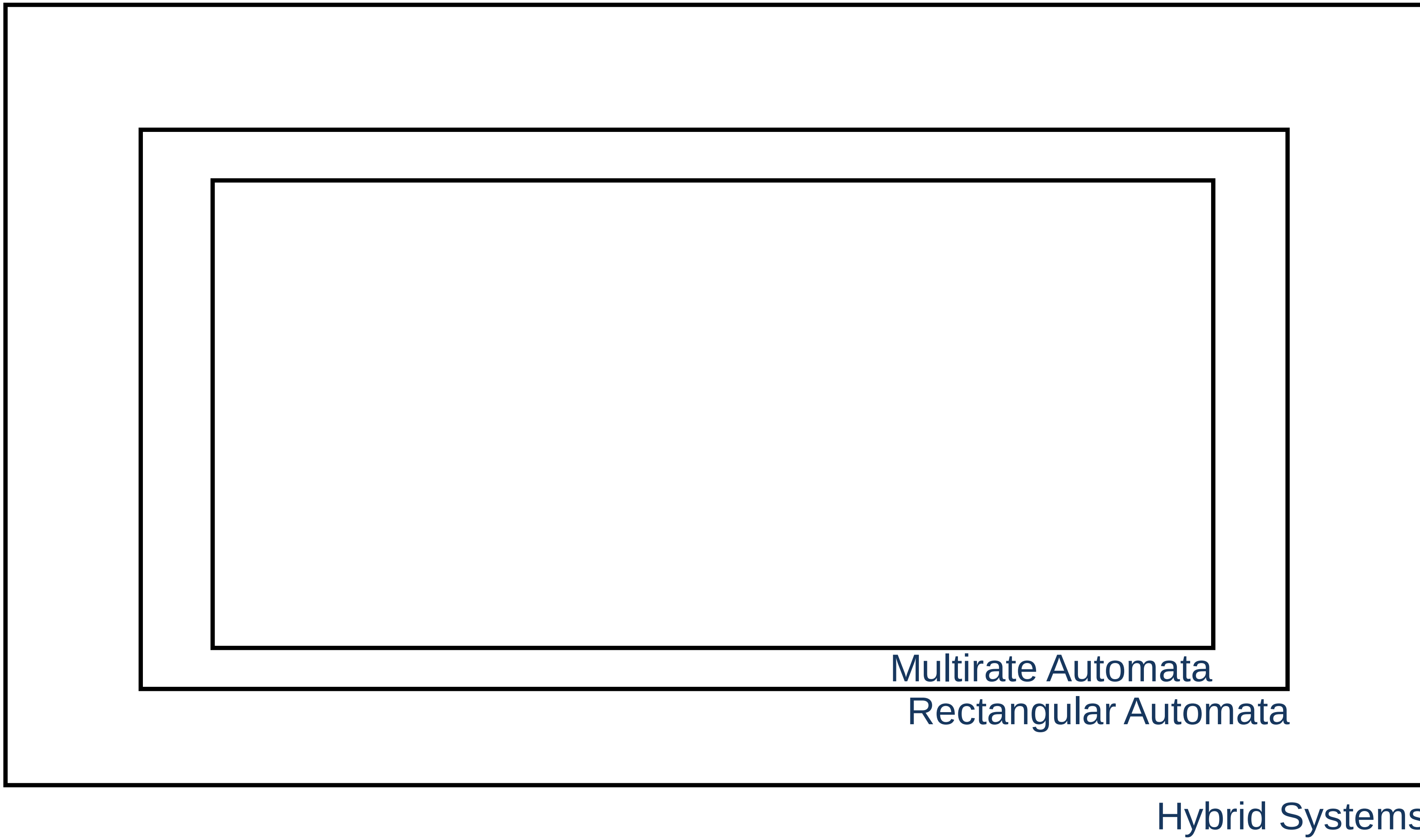

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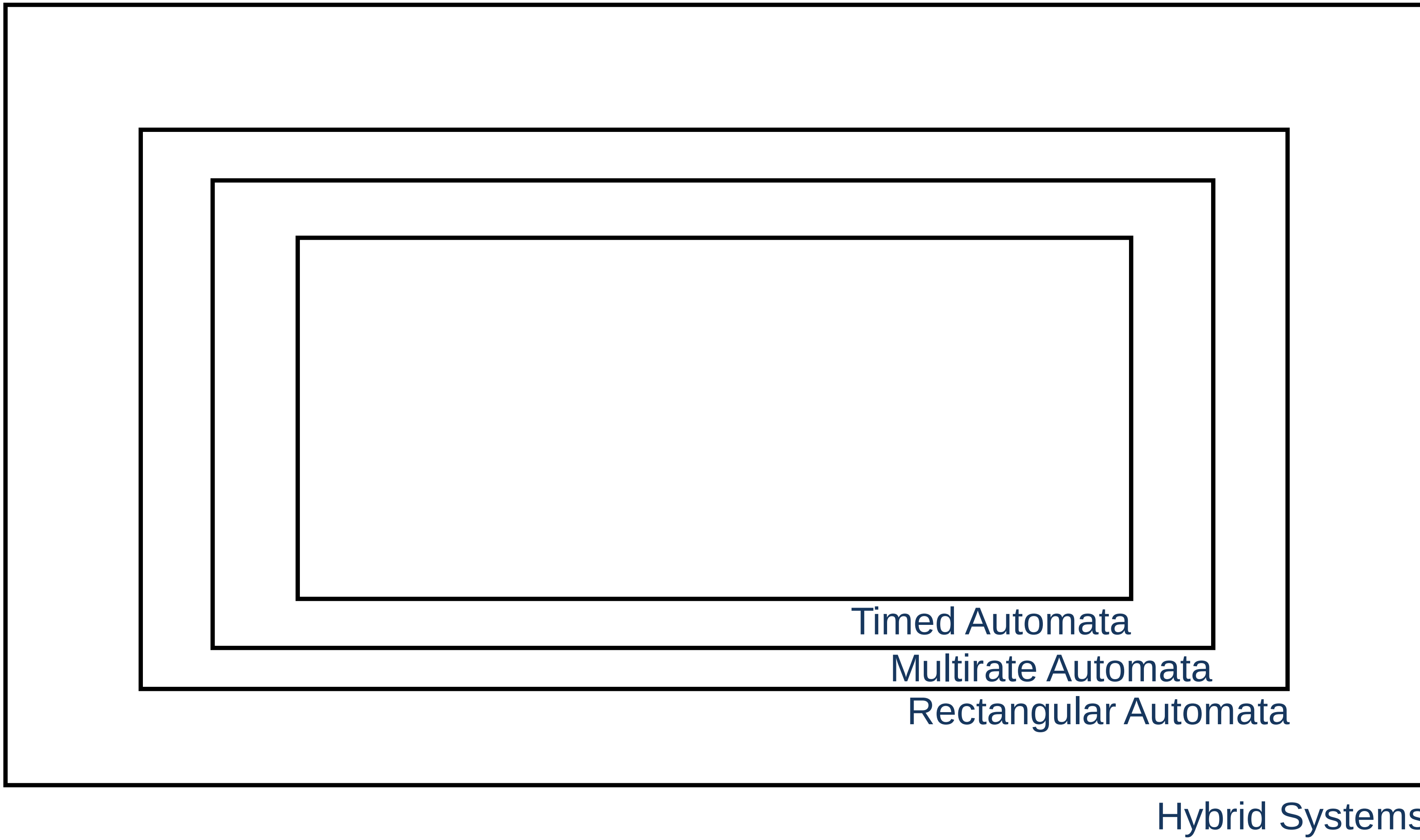

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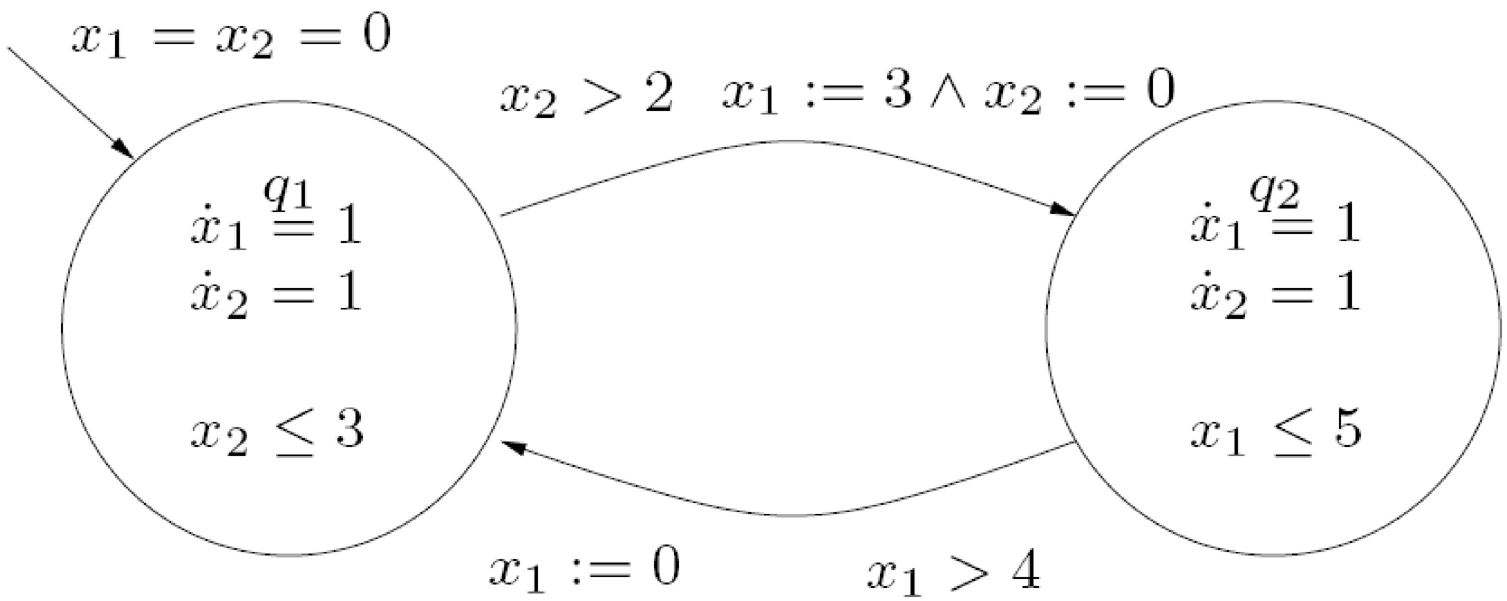
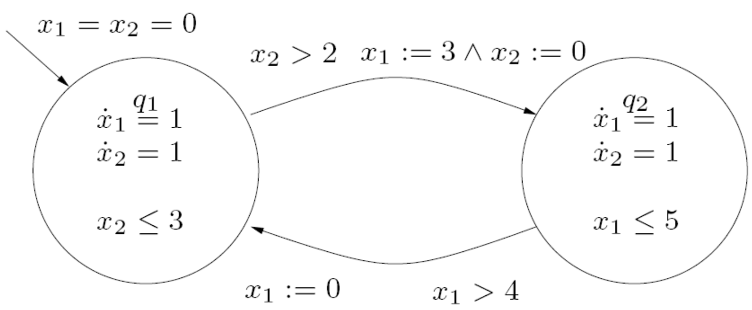


Figure 6.2: Example of a timed automaton.

Timed Automata



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Taken from: John Lygeros, Notes for an ENSIETA short course, February, 2004

Hybrid Automaton

- $Q = \{q_1, q_2\}$;
- $X = \mathbb{R}^2$;
- $f(q_1, x) = f(q_1, x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
- $\text{I}nit = \{(q_1, (0, 0))\}$;
- $\text{D}om(q_1) = \{x \in \mathbb{R}^2 \mid x_2 \leq 3\}$, $\text{D}om(q_2) = \{x \in \mathbb{R}^2 \mid x_1 \leq 5\}$;
- $E = \{(q_1, q_2), (q_2, q_1)\}$;
- $G(q_1, q_2) = \{x \in \mathbb{R}^2 \mid x_2 > 2\}$, $G(q_2, q_1) = \{x \in \mathbb{R}^2 \mid x_1 > 4\}$;
- $R(q_1, q_2, x) = \{(3, 0)\}$, $R(q_2, q_1, x) = \{(0, x_2)\}$

Timed Automata

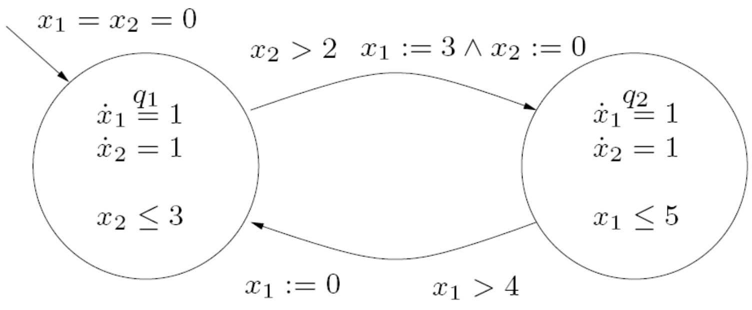

Figure 6.2: Example of a timed automaton.

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I can partition the state space in
- open (possibly infinite) rectangles
- open triangles
- open lines
- points

Timed Automata **admit** Finite Bisimulation!

Region Graph

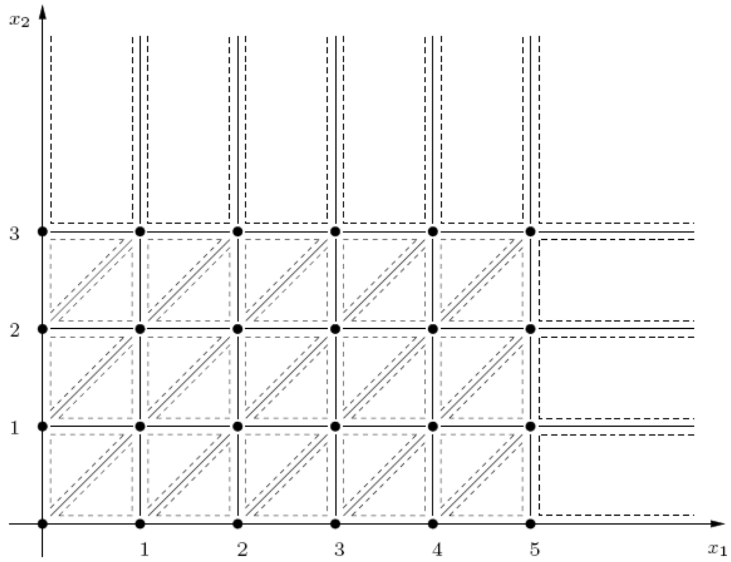


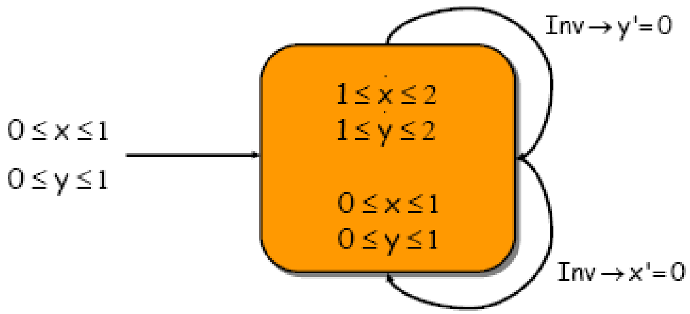
Figure 6.3: Region graph for the timed automaton of Figure 6.2.

Multirate Automata

Inizialized Multirate Automata **admit** Finite Bisimulation!

Rectangular Automata

Inizialized Rectangular Automata **do not admit** Finite Bisimulation!

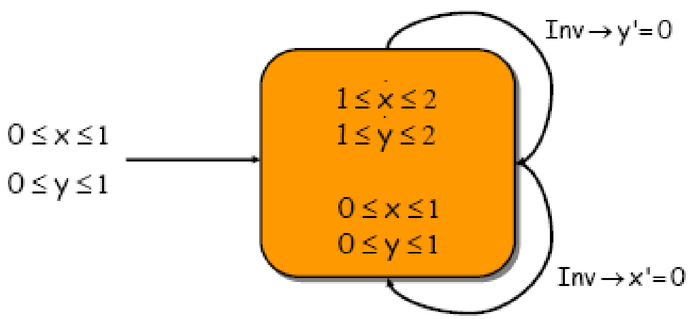


Bisimulation algorithm never terminates

Taken from: Agung Julius, Notes for the course on Hybrid Systems at UPENN, USA

Rectangular Automata

Inizialized Rectangular Automata **do not admit** Finite Bisimulation!



Bisimulation algorithm never terminates

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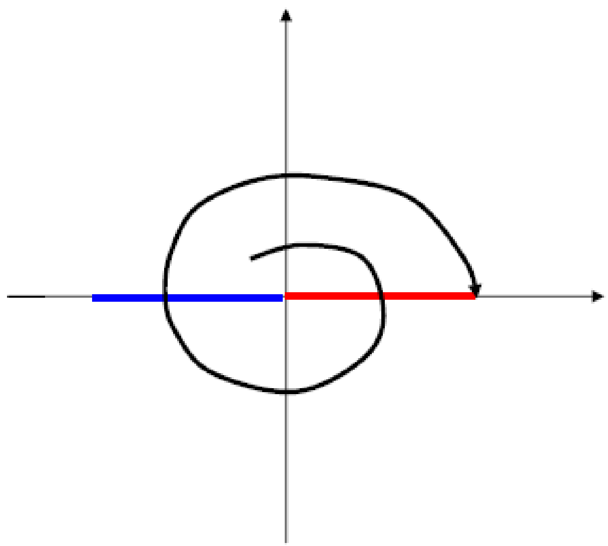
… but

… all initialized rectangular automata admit a finite language equivalence quotient which can be constructed effectively.

… bad news

More complicated dynamics?



Sets

$P_1 = \{(x,0) \mid 0 \le x \le 4\}$

$P_2 = \{(x,0) \mid -4 \le x < 0\}$

$P_3 = R^2 \setminus (P_1 \cup P_2)$

Dynamics

$$\dot{x}_1 = 0.2x_1 + x_2$$

$$\dot{x}_2 = -x_1 + 0.2x_2$$

Bisimulation algorithm never terminates !!

Taken from: Agung Julius, Notes for the course on Hybrid Systems at UPENN, USA

Basic problems

Finite bisimulations of continuous dynamical systems

Given a vector field $F(x)$ and a finite partition of R^n

1. Does there exist a finite bisimulation ?
2. Can we compute it ?

Taken from: Agung Julius, Notes for the course on Hybrid Systems at UPENN, USA

o-minimal hybrid systems

Reminder

Representation issues

- Symbolic representation for infinite sets
- Rectangular sets ? Semi-linear ? Semi-algebraic ?

Operations on sets

- Boolean (logical) operations
- Can we compute Pre and Post ?
- Is our representation closed under Pre and Post ?

Algorithmic termination (decidability)

- No guarantee for infinite transition systems
- We need "nice" alignment of sets and flows
- Globally finite properties

Penn

Taken from: Agung Julius, Notes for the course on Hybrid Systems at UPENN, USA

o-minimal hybrid systems

First-order logic

Every theory of the reals has an associated language

$$(\Re, <, +, -, 0, 1)$$

Universe: $\Re$ — Relation: $<$ — Functions: $+, -$ — Constants: $0, 1$

Variables : $x_1, x_2, x_3, \ldots$

TERMS : Variables, constants, or functions of them

$$x_1 - x_2 + 1, 1 + 1, -x_3$$

ATOMIC FORMULAS : Apply the relation and equality to the terms

$$x_1 + x_2 < -1, 2x_1 = 1, x_1 = x_3$$

(FIRST ORDER) FORMULAS : Atomic formulas are formulas

If φ_1, φ_2 are formulas, then $\varphi_1 \vee \varphi_2, \neg\varphi_1, \forall x.\varphi_1, \exists x.\varphi_1$

Taken from: Agung Julius, Notes for the course on Hybrid Systems at UPENN, USA

o-minimal hybrid systems

First-order logic

Useful languages

$(\Re, <, +, -, 0, 1)$	$\forall x \forall y (x + 2y \geq 0)$
$(\Re, <, +, -, \times, 0, 1)$	$\exists x . ax^2 + bx + c = 0$
$(\Re, <, +, -, \times, e^x, 0, 1)$	$\exists t . (t \geq 0) \wedge (y = e^t x)$

A theory of the reals is **decidable** if there is an algorithm which in a finite number of steps will decide whether a formula is true or not

A theory of the reals admits **quantifier elimination** if there is an algorithm which will eliminate all quantified variables.

$$\exists x . ax^2 + bx + c = 0 \equiv b^2 - 4ac \geq 0$$

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First-order logic

Theory	Decidable ?	Quant. Elim. ?
$(\Re, <, +, -, 0, 1)$	YES	YES
$(\Re, <, +, -, \times, 0, 1)$	YES	YES
$(\Re, <, +, -, \times, e^x, 0, 1)$	?	NO

Tarski's result : Every formula in $(\Re, <, +, -, \times, 0, 1)$ can be decided

1. Eliminate quantified variables
2. Quantifier free formulas can be decided

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O-Minimal Theories

A definable set is $Y = \{(x_1, x_2, \ldots, x_n) \in \Re^n \mid \varphi(x_1, \ldots, x_n)\}$

A theory of the reals is called **o-minimal** if every definable subset of the reals is a finite union of points and intervals

Example: $Y = \{(x) \in \Re \mid p(x) \geq 0\}$ for polynomial p(x)

Recent o-minimal theories

$$(\Re, <, +, -, 0, 1)$$

$$(\Re, <, +, -, \times, 0, 1)$$

$$(\Re, <, +, -, \times, e^x, 0, 1)$$

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Basic answers

Finite bisimulations of continuous dynamical systems

Consider a vector field X and a finite partition of R^n where

1. The flow of the vector field is definable in an o-minimal theory
2. The finite partition is definable in the same o-minimal theory

Then a finite bisimulation always exists.

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